State Operator Correspondence and Entanglement in $\mathrm{AdS}_2/\mathrm{CFT}_1$

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Abstract

Since euclidean global AdS_2 space represented as a strip has two boundaries, the state / operator correspondence in the dual CFT_1 reduces to the standard map from the operators acting on a single copy of the Hilbert space to states in the tensor product of two copies of the Hilbert space. Using this picture we argue that the corresponding states in the dual string theory living on $\mathrm{AdS}_2 \times \mathrm{K}$ are described by twisted version of the Hartle-Hawking states, the twists being generated by a large unitary group of symmetries that this string theory must possess. This formalism makes natural the dual interpretation of the black hole entropy, – as the logarithm of the degeneracy of ground states of the quantum mechanics describing the low energy dynamics of the black hole, and also as an entanglement entropy between the two copies of the same quantum theory living on the two boundaries of global AdS_2 separated by the event horizon.

Contents

1 Introduction and summary

With the help of radial quantization, local operators in a conformal field theory in d dimensions (CFT_d) can be mapped in a one to one fashion to states in the same CFT on $S^{d-1} \times \mathbb{R}$, with $\mathbb{R}$ labelling the time direction. This takes a somewhat trivial form in $d = 1$. Since S^0 is a collection of two points, the states live in the Hilbert space $\mathcal{H} \otimes \mathcal{H}$ of two copies of the CFT_1. On the other hand the absence of spatial separation makes all operators in the Hilbert space $\mathcal{H}$ of a single copy of the CFT_1 local. Thus the state operator correspondence reduces to the standard map between operators $\hat{M}$ in $\mathcal{H}$ and states $\langle a|\hat{M}|b\rangle |a\rangle \otimes |b\rangle$ in the tensor product of two copies of $\mathcal{H}$. In particular the identity operator gets mapped to the maximally entangled state $|a\rangle \otimes |a\rangle$ between the two copies of $\mathcal{H}$.

This picture takes a geometric form for a class of CFT_1 which are dual to string theory on $\mathrm{AdS}_2 \times \mathrm{K}$ for some compact manifold K. These geometries typically arise as the near horizon geometries of black holes in the extremal limit [1, 2]. In this case K contains the compactification manifold as well as the angular coordinates of the asymptotic space-time. When we represent global AdS_2 as an infinite strip, the two copies of the CFT_1 live on the two

boundaries of the strip. Furthermore as argued in [3,4] and reviewed in §2, a single copy of the dual CFT_1 just consists of a finite number (N) of degenerate states representing the ground states of the black hole in a given charge sector. Thus two copies of the CFT_1 living on the two boundaries of AdS_2 will contain N^2 states. By AdS/CFT correspondence [5] we expect the dual string theory on global AdS_2 to also contain N^2 states. One of these states is easy to identify – the Hartle-Hawking vacuum of string theory on $\mathrm{AdS}_2 \times \mathrm{K}$ [6]. This is dual to the identity operator in CFT_1 and hence represents the maximally entangled state between the two copies of the CFT_1. Our goal in this note will be to identify possible origin of the other states in string theory on $\mathrm{AdS}_2 \times \mathrm{K}$ which are expected to exist according to the $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence.

It is generally expected that AdS_2 cannot support any finite energy excitation since this will destroy the asymptotic boundary condition [7].[1] This is not a problem for us since in CFT_1 all states are of the same energy (which we can take to be zero by a shift) and hence we need to look only for zero energy excitations in AdS_2. However this rules out following the usual procedure for constructing excitations in AdS_2 using local fields in the bulk [11,12] since this typically produces finite energy excitations. Some suggestions for constructing zero energy excitations in AdS_2 were made in [7]. However the fragmented geometries of the type discussed in [7] will be absent if the charge carried by the black hole is primitive, since this prevents the total flux to be split into multiple aligned fluxes each through one AdS_2 throat. This still leaves open the possibility of contribution from the scaling solutions described in [13–16] involving three or more throats, with fluxes through different throats aligned along different directions in the charge lattice. But given that the phase space associated with these configurations has finite volume preventing the centers to come arbitrarily close to each other in the quantum theory [17–19], it is more natural to count their effect as part of multi-centered black holes rather than as part of a single AdS_2 throat. In any case in $\mathcal{N} = 4$ supersymmetric string theories there is reasonable evidence that solutions with three or more centers do not contribute to the index [20, 21], and hence we must look for different states.

To look for clues for where the zero energy states might come from, let us examine the state operator correspondence in the dual CFT_1. A linearly independent basis of operators in the CFT_1 is provided by the set of all $N \times N$ hermitian matrices. We shall find it more convenient to work with $N \times N$ unitary matrices instead; if we have sufficiently large number of these

[1] This argument assumes that K is compact. If K contains a non-compact piece *e.g.* $\mathbf{R}^2$, then there is no gap in the spectrum and hence in the infrared limit we can get finite energy excitations. We can use local fields to generate the corresponding states in string theory on in $\mathrm{AdS}_2 \times \mathrm{K}$, leading to non-trivial correlators [8–10].

matrices then any other matrix can be expressed as linear combinations of these matrices. The correlation functions in CFT_1 on S^1 are then traces of products of these matrices. Furthermore since all the N states in CFT_1 are degenerate these $U(N)$ transformations generate exact symmetries of the theory. By $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence this symmetry must be present in the dual string theory as well. Thus to compute these correlation functions in the dual string theory on $\mathrm{AdS}_2 \times \mathrm{K}$ we represent euclidean global AdS_2 as a disk so that the boundary on which CFT_1 lives becomes a circle, and then compute a $U(N)$ twisted partition function in which we require the fields to satisfy a twisted boundary condition along the boundary of AdS_2 [22, 23], the twist being related to the product of the matrices in CFT_1 whose correlation function we wish to compute. This suggests that when we represent AdS_2 as a strip, we can construct the states in string theory on $\mathrm{AdS}_2 \times \mathrm{K}$ via euclidean path integral as in the case of Hartle-Hawking state, albeit with a twisted boundary condition in the asymptotic past. This way the matrix elements between these states naturally produces the twisted partition function.

Formally this prescription gives a complete map between the CFT_1 operators and correlation functions and the corresponding quantities in string theory on $\mathrm{AdS}_2 \times \mathrm{K}$. The main problem of realizing this idea is that at present we do not know of any explicit construction of such $U(N)$ symmetries in string theory on $\mathrm{AdS}_2 \times \mathrm{K}$. However there are special cases where we can realize a small part of this symmetry. Typically as we move around in the moduli spaces of a supersymmetric string theory, we encounter special points at which there are enhanced discrete symmetries (not to be confused with enhanced continuous symmetries). Since typically the black hole microstates get transformed into each other under this discrete symmetry, this has a non-trivial embedding in $U(N)$. The dual string theory on $\mathrm{AdS}_2 \times \mathrm{K}$ also has this symmetry manifest and we can use this to construct the twisted states in AdS_2. While this is far from providing a complete construction of all the states of string theory on $\mathrm{AdS}_2 \times \mathrm{K}$, this at least demonstrates that it is possible to construct non-trivial states in AdS_2 without destroying the asymptotic boundary conditions. To this end we note that even if the near horizon geometry possesses an enhanced discrete symmetry, it need not be a symmetry of the asymptotic theory where the moduli can take different values. Thus our ability to construct these special states is not tied to the existence of some symmetry at infinity that allows us to distinguish different black holes trivially by doing appropriate scattering experiments *e.g.* in [24, 25].

This picture also incorporates naturally the dual interpretation of the entropy of an extremal black hole. It has been known since the work of Bekenstein and Hawking that black holes carry

entropy. One natural explanation of this entropy is that a single black hole represents a large collection of quantum states, and the black hole entropy is given by the logarithm of the degeneracy of microstates the black hole represents. Indeed one of the major successes of string theory has been to reproduce the black hole entropy from the counting of states in the microscopic description of the black hole [26, 27]. On the other hand the geometry of the black hole, which includes a horizon, suggests an alternate interpretation: the black hole entropy represents the result of entanglement between the degrees of freedom living outside the horizon and the degrees of freedom living inside the horizon [28–40].[2] In the framework of AdS_2/CFT_1 correspondence we see that both interpretations are equally good. The black hole entropy $\ln N$ can be interpreted as the logarithm of the degeneracy of a single copy of the CFT_1 living on one of the boundaries of AdS_2, or as the entanglement entropy between the two copies of CFT_1 living on the two boundaries of AdS_2 in the Hartle-Hawking vacuum. Since the latter corresponds to a maximally entangled state, its entanglement entropy is given by $\ln N$.

This observation of course is not new – it is the zero temperature verson of the well known connection between black holes and thermofield dynamics. Given any thermal system, there is a standard doubling trick that allows us to express the thermal averages as quantum mechanical expectation values in an auxiliary system containing two copies of the original Hilbert space [42], and the thermal entropy of the original system can be regarded as the entanglement entropy of the auxiliary system. This correspondence was exploited in [33, 43–46] to identify the two copies of the Hilbert space as being associated with the two boundaries of the extended space-time for a black hole solution. In a related development it was observed in [40] that if we take the global AdS_2 space-time that arises in the near horizon geometry of a black hole in the extremal limit, and calculate the entanglement entropy between the quantum theories living on the two boundaries of this global AdS_2, then in the classical limit the result agrees with the Wald entropy. The argument given above shows that this must continue to hold in the full quantum theory. While we have a prescription for computing the degeneracy of states in the full quantum theory as a partition function of string theory in AdS_2 [3], the prescription of [40] for the holographic computation of the entanglement entropy in CFT_1 involves evaluating the partition function of string theory on a space-time with conical defect. At the classical level the two entropies calculated using these two apparently different computations give the

[2]For a different viewpoint on the relationship between black hole entropy and entanglement see [41] and references therein.

same result, but it is not clear that this equality will continue to hold in the full quantum theory. In §7 we suggest a different approach to computing the entanglement entropy of CFT_1 holographically that does not entail any conical defect and makes the equality of statistical and entanglement entropy manifest even in the quantum theory.

2 CFT_1 and its state operator correspondence

We shall begin by reviewing the properties of the CFT_1 dual to string theory on $\mathrm{AdS}_2 \times \mathrm{K}$ that arises as the near horizon geometry of some extremal black hole. By the usual rules of AdS/CFT correspondence this CFT_1 must be given by the infrared limit of the quantum mechanics describing the dynamics of the brane system producing the black hole. In known examples, *e.g.* the D1-D5-p system producing a five dimensional black hole [26, 27], or the D1-D5-p-KK monopole system producing a four dimensional black hole [47–50], the spectrum of the underlying quantum system has a gap separating the BPS ground states from the first excited states in a fixed charge sector. The gap is small when the charges are large, but is nevertheless non-zero. As a third example consider a BPS black hole in type IIB string theory compactified on a Calabi-Yau 3-fold CY_3, described as a configuration of 3-brane wrapped on an appropriate supersymmetric three cycle inside CY_3. The quantum mechanics describing the system is a (0+1) dimensional sigma model with the moduli space of supersymmetric 3-cycles as target space. Again as long this moduli space is compact we expect the spectrum of the quantum theory to be discrete, and there will be a gap between the supersymmetric ground states and the first excited state. We shall assume that this is always the case for the quantum system describing an extremal black hole. Then in the infrared limit only the ground states of this quantum mechanics will survive, and the CFT_1 will consist of a finite number N of degenerate states.

The usual state operator correspondence in a d dimensional conformal field theory relates every local operator in the conformal field theory to a state in the conformal field theory on $S^{d-1} \times \mathbb{R}$. For $d \geq 2$ this is usually achieved by the standard map from S^d to $S^{d-1} \times \mathbb{R}$ via the coordinate transformation that takes the north and the south poles of S^d to $\pm\infty$ of $\mathbb{R}$. In this case local operators inserted at the south pole of S^d create the corresponding states at $\tau = -\infty$ on $S^{d-1} \times \mathbb{R}$. The state operator correspondence in $d = 1$ works in a more or less

similar way. First the map from S^1 to $S^0 \times \mathbb{R}$ is achieved via the coordinate transformation

$$\sigma + i\tau = 2\tan^{-1}\tanh\left(\frac{i\theta}{2}\right) . \tag{2.1}$$

Indeed this takes the circle labelled by θ to a pair of lines $S^0 \times \mathbb{R}$ where S^0 corresponds to the pair of points $\sigma = 0, -\pi$ and $\mathbb{R}$ is labeled by τ. The points $\theta = \pm\pi/2$ are mapped to $\tau = \pm\infty$, the segment $-\pi/2 < \theta < \pi/2$ is mapped to the line at $\sigma = 0$ and the segment $\pi/2 < \theta < 3\pi/2$ is mapped to the line at $\sigma = -\pi$. Thus CFT$_1$ on $S^0 \times \mathbb{R}$ actually corresponds to two copies of the CFT$_1$. On the other hand since for $d = 1$ there is no notion of spatial separation, every operator acting on the Hilbert space $\mathcal{H}$ of a single copy of the CFT$_1$ can be regarded as a local operator. Thus we are looking for a map between the set of operators acting on a single copy of $\mathcal{H}$ to the set of states living on two copies of $\mathcal{H}$ at the two boundaries $\sigma = 0, -\pi$. It is straightforward to construct such a map, – the operator $\hat{M}$ inserted at $\theta = -\pi/2$ on S^1 creates the state

$$|M\rangle\rangle = M_{ab}\,|a\rangle_{(1)} \otimes |b\rangle_{(2)} , \qquad M_{ab} \equiv \langle a|\hat{M}|b\rangle , \tag{2.2}$$

on $S^0 \times \mathbb{R}$ at $\tau = -\infty$. Here $\{|a\rangle\}$ denotes a complete set of orthonormal basis states in $\mathcal{H}$, the subscripts $_{(1)}$ and $_{(2)}$ denote the two copies of $\mathcal{H}$, and $|\ \rangle\rangle$ denotes a state in $\mathcal{H} \otimes \mathcal{H}$. For this state the density matrix in the Hilbert space of the first copy of CFT$_1$, obtained by tracing over the states in the second copy, is given by

$$(MM^\dagger)_{ac}\,|a\rangle\langle c| . \tag{2.3}$$

Given two such states $|M\rangle\rangle$ and $|P\rangle\rangle$, we have:

$$\langle\langle M|P\rangle\rangle = M^*_{ab}P_{ab} = Tr(M^\dagger P) . \tag{2.4}$$

This can be interpreted as the two point function of $\hat{M}^\dagger$ and $\hat{P}$ in CFT$_1$ on S^1, in accordance with the usual rules of state operator correspondence in conformal field theories.

A special state corresponding to the identity operator in the CFT$_1$ is

$$|I\rangle\rangle = |a\rangle_{(1)} \otimes |a\rangle_{(2)} . \tag{2.5}$$

We shall refer to this state as the vacuum state although all states have equal energy. The corresponding density matrix is $|a\rangle\langle a|$, showing that it is a maximally entangled state. This however is not the only maximally entangled state, – it follows from (2.3) that for any unitary operator $\hat{W}$ the corresponding state $|W\rangle\rangle$ has density matrix $|a\rangle\langle a|$, and hence describes a

maximally entangled state. Furthermore (2.4) shows that for unitary operators W and V, the inner product $\langle\langle W|V\rangle\rangle$ is given by $Tr(W^{-1}V)$.

Note that in CFT_1, $Tr(W)$ may be expressed as

$$\langle\langle I|W_{(1)}|I\rangle\rangle\,, \tag{2.6}$$

where $W_{(1)}$ denotes the operator W acting on the first copy of the Hilbert space. Thus CFT_1 correlation functions can be interpreted as the expectation values of the operators acting on the first copy of the CFT_1 in the vacuum state.

Our goal will be to seek possible representation of these states in dual string theory on $\mathrm{AdS}_2 \times \mathrm{K}$.

3 AdS_2 space in different coordinates

In this section we shall review some facts about the near horizon geometry of black holes in the extremal limit. For higher dimensional branes one usually takes a brane solution at zero temperature and then takes the near horizon limit to get an AdS space-time. This corresponds to looking at excitations whose energies are small from the point of view of the asymptotic observer but large compared to the temperature of the brane. This is not a sensible limit for a black hole since, as reviewed in §2, the black hole quantum mechanics has a gap that separates the ground state from the first excited state, and so the only low energy excitations are zero energy excitations. So the sensible infrared limit is to take the energy scale to zero as we take the temperature to zero.[3] This can be achieved by taking the extremal limit in an appropriately rescaled coordinate system in which the two horizons remain finite coordinate distance away from each other [3, 51]. In this limit part of the near horizon geometry of the black hole involving the time and the radial coordinates takes the form [7, 52, 53]

$$ds^2 = a^2\left[-(r^2-1)dt^2 + \frac{dr^2}{r^2-1}\right]\,, \tag{3.1}$$

where a is some constant. Here, up to a rescaling, r and t can be identified as the radial and the time variables of the full black hole solution. The inner and the outer horizons are at $r = \pm 1$. The metric (3.1) describes a locally AdS_2 space-time. This can be extended to global AdS_2 with the help of the coordinate transformation [7]:

$$T \pm \sigma = 2\tan^{-1}\tanh\frac{1}{2}\left(t \pm \frac{1}{2}\ln\frac{r-1}{r+1}\right)\,. \tag{3.2}$$

[3] I wish to thank Hong Liu for a discussion on this point.

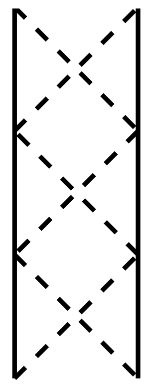

Figure 1: Global AdS_2 and the location of the horizon(s). The two vertical solid lines label the two boundaries of AdS_2 at $\sigma = -\pi$ (left) and $\sigma = 0$ (right). The dashed lines label the locations of the event horizons of the original black hole.

In this coordinate system the metric takes the form:

$$ds^2 = \frac{a^2}{\sin^2 \sigma}(-dT^2 + d\sigma^2)\,. \tag{3.3}$$

The range of (T, σ) can be taken to be $(-\pi < \sigma < 0,\ -\infty < T < \infty)$. This space has two boundaries, at $\sigma = 0$ and at $\sigma = -\pi$. These two boundaries lie on opposite sides of the horizon $r = \pm 1$ of the original metric (3.1). The asymptotic boundary $r \to \infty$ in the original metric (3.1) lies at $\sigma = 0$. Fig.1 shows AdS_2 in the (σ, T) coordinate system where the locations of the horizons at $r = \pm 1$ have been shown by the dashed line [7, 54].

Let us now consider the euclidean version of the metrics (3.1) and (3.3). The euclidean version of the metric (3.1) is obtained by replacing t by $-i\theta$. This gives the metric

$$ds^2 = a^2 \left[(r^2 - 1)d\theta^2 + \frac{dr^2}{r^2 - 1} \right] . \tag{3.4}$$

Introducing new coordinate $\rho = \sqrt{(r-1)/(r+1)}$ we can express the metric as

$$ds^2 = \frac{4\,a^2}{(1-\rho^2)^2} \left[d\rho^2 + \rho^2 d\theta^2 \right] . \tag{3.5}$$

In this coordinate it is clear that absence of conical singularity at $\rho = 0$ ($r = 1$) requires θ to be a periodic coordinate with period 2π. The resulting two dimensional space spanned by (ρ, θ) with $0 \le \rho < 1$, $\theta \equiv \theta + 2\pi$ describes a unit disk.

Euclidean version of the metric (3.3) is obtained by replacing T by $-i\tau$. This gives

$$ds^2 = \frac{a^2}{\sin^2 \sigma}(d\tau^2 + d\sigma^2)\,. \tag{3.6}$$

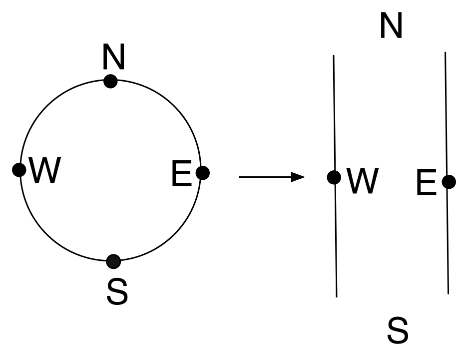

Figure 2: Conformal map from unit disk to the strip. The left boundary of the strip is at $\sigma = -\pi$ and the right boundary is at $\sigma = 0$.

This time there is no periodicity requirement of τ. This describes an infinite strip spanned by (τ, σ) with $-\infty < \tau < \infty$, $0 < \sigma < \pi$.

Even though for the Lorentzian signature the coordinates (r, t) for $r \geq 1$ cover only a patch of the global AdS_2 spanned by the coordinates (T, σ) in (3.3), the euclidean spaces (3.4) and (3.6) have an exact one to one map:

$$\sigma + i\tau = 2\tan^{-1}\tanh\frac{1}{2}\left(\frac{1}{2}\ln\frac{r-1}{r+1} + i\theta\right) = 2\tan^{-1}\tanh\frac{1}{2}\left(\ln\rho + i\theta\right) . \tag{3.7}$$

This is the standard one to one conformal map between the unit disk and the infinite strip as shown in Fig. 2. The segment of the boundary of the disk $\left(\rho = 1, -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$ gets maps to the $\sigma = 0$ boundary of the strip, and the segment $\left(\rho = 1, \frac{\pi}{2} < \theta < \frac{3\pi}{2}\right)$ gets mapped to the $\sigma = -\pi$ boundary of the strip. In fact for $\rho = 1$ this reduces to the standard map from S^1 to $S^0 \times \mathbb{R}$ described in (2.1).

4 $\mathbf{AdS_2/CFT_1}$ correspondence

Next we shall review some aspects of the AdS_2/CFT_1 correspondence proposed in [3]. Consider an $AdS_2 \times K$ geometry arising as the near horizon limit of an extremal black hole carrying a fixed charge. Then this is dual to the CFT_1 obtained as the infrared limit of the brane system describing the dynamics of the black hole. If CFT_1 has N states, then the black hole entropy, identified as the logarithm of the number of states of the CFT_1, is given by $\ln N$.[4]

[4]If the black hole solution has hair modes then we must remove their contribution while counting N [55,56].

An algorithm for computing this entropy using the bulk description was given in [3, 4]. For this we consider the euclidean $\mathrm{AdS}_2 \times \mathrm{K}$ geometry given in (3.4) and denote by $\widehat{Z}_{AdS_2}$ the partition function of string theory in $\mathrm{AdS}_2 \times \mathrm{K}$, computed with the natural boundary condition that requires us to fix the electric fields at infinity and integrate over the r independent modes of the gauge fields.[5] Due to infinite size of the euclidean AdS_2 space this partition function is divergent, so we need to regularize the divergence by putting a cut-off on r, say $r \le r_0$ or equivalently $\rho \le 1-\epsilon$. This makes AdS_2 have a finite volume and the boundary of AdS_2, situated at $r = r_0$, have a finite length which we shall call L. Now by AdS/CFT correspondence $\widehat{Z}_{AdS_2}$ should be given by the partition function of the CFT_1 living on the boundary circle at $r = r_0$. The latter in turn is given by $Tre^{-LH} = N\, e^{-LE_0}$, where H is the Hamiltonian of CFT_1 and E_0 is the energy of the N degenerate states of CFT_1. Thus we have

$$\widehat{Z}_{AdS_2} = N\, e^{-E_0 L}\,. \tag{4.1}$$

This suggests that in order to calculate N, we first calculate $\widehat{Z}_{AdS_2}$ and then extract its finite part by expressing it as $d_{hor}\, e^{CL}$ for some finite constants d_{hor} and C in the $L \to \infty$ limit. In that case C can be identified with $-E_0$ and d_{hor}, called the quantum entropy function, can be identified as the ground state degeneracy N of the black hole. This gives a complete prescription for computing the black hole entropy in the bulk theory. In the classical limit d_{hor} defined this way reproduces the exponential of the Wald entropy [3]. In principle quantum corrections may be computed directly [57–59], or, for supersymmetric black holes, using localization [60]. Significant advances towards computing d_{hor} using localization techniques have been made recently [61].

By adjusting the boundary terms in the action describing string theory on $\mathrm{AdS}_2 \times \mathrm{K}$ we can make the constant C vanish, so that $\widehat{Z}_{AdS_2}$ can be directly identified as the degeneracy of CFT_1. This corresponds to a constant shift in the definition of the energy of CFT_1 to make E_0 vanish. For simplifying the notation we shall proceed with this convention although we can always include the explicit E_0 dependence in all the equations below if so desired.

Can we use the bulk description to calculate other observables in CFT_1? Since CFT_1 consists of a finite number of degenerate states, the only observables are $N \times N$ matrices M acting on this N dimensional vector space. Let us focus on the cases of unitary matrices which generate $U(N)$ transformations in this N dimensional vector space. Since the all N states in

[5] As discussed in [3], this requires introducing a Wilson loop operator along the boundary of AdS_2 while computing $\widehat{Z}_{AdS_2}$.

CFT_1 are degenerate, $U(N)$ is an exact symmetry of CFT_1. Hence it must also exist as an exact symmetry of the dual string theory on $\mathrm{AdS}_2 \times \mathrm{K}$, and corresponding to any $U(N)$ element W there must be a corresponding transformation (also denoted by W) acting on the variables in the dual string theory.[6] Computing $Tr(W)$ in CFT_1 will then correspond to evaluating the partition function of string theory on $\mathrm{AdS}_2 \times \mathrm{K}$ with a W twisted boundary condition on the bulk fields under $\theta \to \theta + 2\pi$.

While this gives a way to relate $Tr(W)$ in the boundary theory to a specific quantity in the bulk theory, in general the action of W on the bulk fields is not known. But in special cases, *e.g.* when W represents a known $\mathbb{Z}_k$ symmetry generator of the theory, there is a natural lift of the action of W to the bulk fields. In this case $Tr(W)$ in CFT_1 can be identified as a $\mathbb{Z}_k$ twisted partition function in the bulk theory. Since the boundary circle of the euclidean AdS_2 space described by the metric (3.5) is contractible in the interior, a W twisted boundary condition is not allowed there. Thus the original $\mathrm{AdS}_2 \times \mathrm{K}$ geometry does not contribute to this amplitude. But a $\mathbb{Z}_k$ orbifold of the $\mathrm{AdS}_2 \times \mathrm{K}$ geometry does contribute and gives a non-zero answer for $Tr(W)$ [22]. This prescription has passed non-trivial tests in a class of $\mathcal{N} = 4$ supersymmetric string theories where an independent microscopic computation of $Tr(W)$ is possible [22, 23].[7] Note that for $W = 1$ we recover the original partition function $\widehat{Z}_{AdS_2}$ on the bulk side and $Tr(1) = N$ on the CFT_1 side.

5 States in string theory on AdS_2

The result of the previous sections suggests a way of associating the states in two copies of CFT_1 with states in string theory on $\mathrm{AdS}_2 \times \mathrm{K}$. Let us for definiteness work with states in CFT_1 on $S^0 \times \mathbb{R}$ of the form $|W\rangle\rangle$ with unitary operators W. To the states $|W\rangle\rangle$ and $|V\rangle\rangle$ we want to associate wave-functions f_W and f_V in string theory such that the inner product of f_W

[6] In AdS/CFT correspondence global symmetries in the boundary theory arise as local symmetries of the bulk theory. We shall not try to make this distinction here since we shall use the $U(N)$ transformations to twist the boundary condition, and for this only the global part of the group is relevant anyway. However it is important that there should not be any dynamical $U(N)$ gauge fields in the bulk since this will force the black hole to carry fixed charges under the Cartan generators of $U(N)$ [3].

[7] Although the computation of $Tr(W)$ in the bulk theory will be the same as the one described in [22], the spirit in which we want to use this is different. In [22, 23] the asymptotic moduli were adjusted to also have the unbroken $\mathbb{Z}_k$ symmetry so that we could compute $Tr(W)$ microscopically and compare with the macroscopic result. In contrast, here we want to interprete $Tr(W)$ as an observable in the near horizon theory irrespective of whether or not it is a symmetry of the asymptotic theory. For this we only need to adjust the asymptotic moduli to be in a certain subspace which under attractor flow [62–64] approaches the point of enhanced discrete symmetry at the horizon.

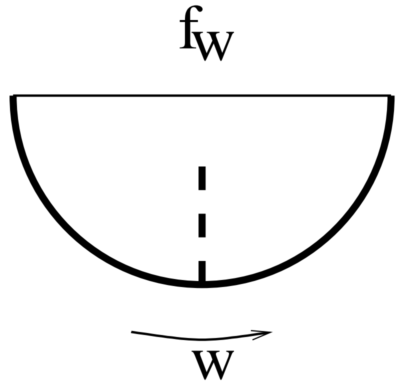

Figure 3: Generating a state in string theory on AdS_2 from a state $W_{ab}|a\rangle_{(1)}|b\rangle_{(2)}$ in the two copies of the Hilbert space of CFT_1. The thick semi-circular line is the boundary of AdS_2 whereas the thin diameter is the line on which the string fields, appearing in the argument of f_W, live. The dashed line reaching the boundary of AdS_2 denotes a cut which relates the field configurations on the right of the cut to those on the left of the cut by a transformation by W.

and f_V generates the CFT_1 two point function $Tr(W^{-1}V)$. The latter in turn is described by the path integral of string theory in $\mathrm{AdS}_2 \times \mathrm{K}$, with the boundary condition that as $\theta \to \theta + 2\pi$ near the boundary, the fields are twisted by $W^{-1}V$. This can be achieved by defining f_W to be generated by the result of string theory path integral over the half disk (or semi-infinite strip) with a cut corresponding to the transformation W that reaches the boundary of the half disk (see Fig. 3). The inner product $\langle\langle W|V\rangle\rangle$ will then be obtained by gluing the two half disks, with cuts W and V along the boundaries, along their common diameter. This is given by the path integral over the whole disk with a twisted boundary condition by $W^{-1}V$ along the boundary, as required.

Note that in Fig. 3 the cut can either end in the interior of the half disk if it is an allowed configuration in string theory, or reach the diameter on which the wave-function f_W is defined. This is reminiscent of the sum over geometries in [33]. The necessity for allowing the cut to reach the diameter can be seen in the computation of $\langle\langle W|W\rangle\rangle$. The dominant contribution to this amplitude somes from the configuration where the cut extends all the way across the disk, representing the usual $\mathrm{AdS}_2 \times \mathrm{K}$ geometry without any twist.

While this gives an abstract prescription for associating the states living in two copies of the Hilbert space of CFT_1 to states of string theory on $\mathrm{AdS}_2 \times \mathrm{K}$, in general we cannot explicitly construct these states since the action of W on the bulk fields is not known. However for the special cases when W can be associated with some known $\mathbb{Z}_k$ symmetry generator of string

theory on $\mathrm{AdS}_2 \times \mathrm{K}$, we can explicitly construct the corresponding state; the path integral will be over all field configurations whose boundary values jump by the action of this discrete symmetry transformation W at some point on the boundary. Even though this is a special case, this at least demonstrates that it is possible for global AdS_2 to admit non-trivial quantum states.

It is worth emphasizing that this discrete symmetry need not be a symmetry away from the horizon – the asymptotic moduli could be in a configuration that breaks this symmetry while the attractor mechanism pulls the near horizon geometry towards a configuration that is invariant under this symmetry (see footnote 7). From this point of view this discrete symmetry is on the same footing as the rest of the proposed $U(N)$ symmetry, in that this symmetry is present in the near horizon geometry but not necessarily present away from the horizon.

6 Conformal invariance of the correlation functions

AdS_2 space has $\mathrm{SL}(2, \mathbb{R})$ as a global isometry. Thus we expect the correlation functions in CFT_1 to be $\mathrm{SL}(2, \mathbb{R})$ invariant. We shall now see how this is manifest in the formalism described above. If we take a set of $U(N)$ elements $W_1, \cdots W_n$ then their correlation function in CFT_1 will be given by $Tr(W_1 \cdots W_n)$. This can be interpreted as the n-point correlation function on the euclidean time circle, but since all the states have zero energy the correlation function depends only on the cyclic time ordering of the operators and not on the explicit time coordinates where the operators are inserted. In the bulk description this is given by a partition function in which the boundary values of the fields are twisted by successive applications of $W_1, \cdots W_n$. Again the correlation function depends on the order in which the W_i twists are applied, but not where we put the cut corresponding to the transformation W_i. Now under an $\mathrm{SL}(2, \mathbb{R})$ transformation the boundary circle gets mapped to itself in a one to one fashion, and the cyclic order of any set of points on the boundary is preserved. Thus the correlation function is manifestly invariant under $\mathrm{SL}(2, \mathbb{R})$ transformation.

The full conformal group in one dimensions is in fact much bigger, given by the full Virasoro group $\mathrm{Diff}(S^1)$ that maps the circle to itself in a one to one fashion. This also preserves the cyclic ordering of the points on the circle and hence is a symmetry of the correlation function in CFT_1. To see how this comes about in string theory on $\mathrm{AdS}_2 \times \mathrm{K}$ note that we can find a group of diffeomorphisms in AdS_2 satisfying the conditions:

1. The elements of the group approach $\mathrm{Diff}(S^1)$ as we approach the boundary.

2. They preserve the asymptotic boundary conditions on the metric and various other fields in $AdS_2 \times K$ [65,66] (see [67] for explicit forms of these diffeomorphisms near the boundary).

3. They are well defined in the interior of AdS_2.

In other words there is an asymptotic symmetry of euclidean AdS_2 corresponding to the group $\mathrm{Diff}(S^1)$ even though this is not an isometry of AdS_2. Since in computing $Tr(W_1 \cdots W_n)$ in the bulk theory we integrate over all fields subject to the asymptotic boundary conditions, the path integral will automatically include orbits of the above group of transformations. Furthermore these transformations will not affect the order in which the cuts are arranged along the boundary. Thus correlations functions computed in the bulk theory with twisted boundary conditions will also be manifestly invariant under the full $\mathrm{Diff}(S^1)$ group.

7 Entanglement vs statistical entropy

When W is the identity operator then there is no cut on the disk, and the corresponding state in string theory is simply the Hartle Hawking state obtained by string theory path integral over the half disk without any cut. In the boundary theory this represents the state $|I\rangle\rangle$ defined in (2.5).[8] Thus the Hartle-Hawking state is the maximally entangled state in the two copies of the CFT_1 living on the two boundaries of global AdS_2. The corresponding entanglement entropy is given by

$$S_{entangle} = \ln N \,, \tag{7.1}$$

where N is the dimension of the Hilbert space of CFT_1. This agrees with the statistical entropy defined as the logarithm of the degeneracy of states in the CFT_1. This is a special case of the general result of [33], and confirms the explicit finding of [40] that the entropy of an extremal black hole can be interpreted as the entanglement entropy between the two copies of CFT_1 living on the two boundaries of the global AdS_2. In fact not only the state (2.5) has entanglement entropy $\ln N$, any twisted state $|W\rangle\rangle$ introduced earlier has density matrix $W^\dagger W = I$ and hence entanglement entropy $\ln N$.

The equality between the statistical and entanglement entropy is obvious in the CFT_1. This can also be seen in the bulk theory as follows. The standard procedure for computing the

[8]This is precisely the structure of the entangled state found in [40] for the special case of extremal BTZ black holes by starting with a finite temperature system and then taking its zero temperature limit.

entanglement entropy in a CFT uses the definition:

$$S_{ent} = -\lim_{n\to 1} \frac{d}{dn} \frac{Tr(\rho^n)}{(Tr\rho)^n}\,. \tag{7.2}$$

Thus for computing S_{ent} using the bulk description we need to find a way of computing $Tr(\rho^n)$ holographically. This can be done as follows. Since the CFT_1 lives on a pair of points and since the state we are interested in is the Hartle-Hawking state, we can construct this as a path integral in the CFT_1 on a line of length $L/2$ with boundary conditions ϕ_1 and ϕ_2 at the two ends labelling the states of the two CFT_1's. Eventually we want to take $L\to\infty$ (although since all states have the same energy the L dependence is trivial). Now to compute the unnormalized density matrix $\rho(\phi_1,\phi_1')$ by taking the trace over states of the second system we take another copy of this line segment with boundary conditions ϕ_1' and ϕ_2' and glue the second ends of the two segments after identifying ϕ_2 with ϕ_2'. This leaves us with a line segment of length L with boundary conditions ϕ_1 and ϕ_1' at the two ends. For computing ρ^n we simply take n copies of this line segment and glue the primed end of the i-th segment with the unprimed end of the $(i+1)$-th segment for $1\le i\le (n-1)$. This gives a line segment of length $n\,L$ with boundary conditions ϕ_1 and ϕ_n' at the two ends. Finally to calculate $Tr(\rho^n)$ we join the two ends of this line segment by identifying ϕ_1 with ϕ_n' and carry out the path integral over the fields of the CFT_1 on this circle. Thus the holographic computation of $Tr(\rho^n)$ will involve calculating the partition function of string theory over all spaces each of which has a boundary circle of length nL and approaches the AdS_2 geometry asymptotically. If we denote this contribution by $\widehat Z_{AdS_2}(n)$ then we have

$$Tr(\rho^n) = \widehat Z_{AdS_2}(n)\,. \tag{7.3}$$

Hence from (7.2) we get

$$S_{ent} = -\lim_{n\to 1} \frac{d}{dn} \frac{\widehat Z_{AdS_2}(n)}{\widehat Z_{AdS_2}(1)^n}\,. \tag{7.4}$$

$\widehat Z_{AdS_2}(1)$ can be identified with the $\widehat Z_{AdS_2}$ given in (4.1). To compute $\widehat Z_{AdS_2}(n)$ we note that the leading contribution to this partition function comes from the euclidean AdS_2 geometry (3.4) itself with the cut-off on r adjusted to produce the appropriate boundary length nL. Even the full quantum contribution to the partition function will be given by the quantum contribution to $\widehat Z_{AdS_2}$ with a different infrared cut-off so that the boundary has length nL. Thus we have from (3.4)

$$\widehat Z_{AdS_2}(n) = N e^{-n\,E_0\,L}\,. \tag{7.5}$$

Substituting this into (7.4) gives

$$S_{ent} = -\lim_{n\to 1}\frac{d}{dn}N^{1-n} = \ln\, N\,, \tag{7.6}$$

which is manifestly equal to the statistical entropy.

Note that this approach differs from the holographic prescription of [40, 68] for the computation of $Tr(\rho^n)$. In the latter approach while computing $\widehat{Z}_{AdS_2}(n)$ we compute the partition function of of string theory on n-fold cover of AdS$_2$. This has a conical defect at the center. On the other hand in our approach we first take an n-fold cover of the boundary circle and then integrate over all possible bulk space-time with this boundary condition. The leading saddle point is the AdS$_2$ space itself with a different infrared cut-off. Although classically the two approaches give the same result, it is not clear *a priori* if the agreement will continue to hold in quantum string theory. Indeed at present we do not know how to define string theory on a space with conical defect for an arbitrary defect angle. Clearly understanding the relationship between these two computations will be desirable since it might also given us a clue as to when, why and how the holographic prescription [40, 68–70] for computing entanglement entropy works.

8 Information loss problem

Since an extremal black hole has zero temperature, we cannot directly formulate the usual information loss puzzle involving absorption and subsequent Hawking radiation from such a black hole unless we allow the black hole to go through an intermediate non-extremal state. The best we can do is the following. Suppose we probe the extremal black hole by an external agent carrying energy lower than the gap that separates the ground state of the black hole from the first excited state. In this case the only possible transitions are to the other ground states of the black hole. Will the black hole retain a memory of this probe that can be tested using a second experiment with another probe? To address this question we note that the effect of such probes can be described by (linear combinations of) the twist operators described earlier.[9] If

[9] In general the relation between the external probe and the twist operator will be complicated and depend on the interpolating geometry that connects the near horizon region to the asymptotic region. For example in the case of the $\mathbb{Z}_k$ symmetry which represents a known discrete symmetry of string theory at special points in the moduli space, the relation between the twist operators and external probes depends on how a low energy probe at infinity where the $\mathbb{Z}_k$ symmetry may be broken, transforms itself into a linear combination of $\mathbb{Z}_k$ eigenstates by the time it reaches the horizon where the $\mathbb{Z}_k$ symmetry is unbroken. Such a relation can in principle be derived from the knowledge of the full black hole solution that interpolates between the near horizon geometry with unbroken $\mathbb{Z}_k$ symmetry and the asymptotic region.

we now perform two successive experiments on the black hole, one with the probe W followed by another with probe V, then the effect of the first probe on the second experiment will be tested by the two point function of W and V. Since in general this two point function is non-zero, we see that the black hole does retain the memory of the first probe. The simplest example of this is the case where WV represents a $\mathbb{Z}_k$ twist of the type discussed in [22,23]. In this case the leading saddle point that contributes to this correlation function is a $\mathbb{Z}_k$ orbifold of $\mathrm{AdS}_2 \times \mathrm{K}$, and has a contribution of order $N^{1/k}$ [22] compared to the contribution of order N to $Tr(1)$. Thus as in the case of [33], the non-vanishing contribution comes only after we sum over non-trivial saddle points, and is suppressed by a power of N.

9 Speculations on the enhanced symmetry

If our proposal for the state operator correspondence is correct, this will imply that string theory in the near horizon geometry of an extremal black hole has a $U(N)$ symmetry by which we can twist the boundary conditions on the fields. Since N can be very large this implies a large group of symmetries of the theory. This is expected to be symmetry of string theory in the $\mathrm{AdS}_2 \times \mathrm{K}$ geometry but not of the string theory in asymptotically flat space time in which the black hole is embedded, since the $U(N)$ symmetry acting on the N degenerate states of the black hole becomes an exact symmetry only in the infrared limit in which there is a decoupled quantum mechanics of the N degnerate BPS states.

At present we do not have any concrete understanding of how such enhanced symmetries could arise. We shall end by making some random observations:

1. In the classical limit the black hole entropy $\ln N$ goes to infinity. Thus if the $U(N)$ symmetry is present in the classical limit, then it must appear as a $U(\infty)$ symmetry which is broken down to $U(N)$ by quantum effects. Since $U(\infty)$ is a symmetry of the infinite dimensional complex grassmannians, one could wonder if grassmannians might play a role in string theory on $\mathrm{AdS}_2 \times \mathrm{K}$. Alternatively the $U(N)$ symmetry could arise only as a symmetry of the quantum theory with no classical analog.

2. Typically for a BPS black hole in a supersymmetric theory carrying a fixed charge, some of the moduli scalar fields are fixed at the horizon due to the attractor mechanism, but the other moduli may remain free and label the moduli space of the near horizon geometry. As we move around in this moduli space, the discrete symmetries of the theory may

change, being either non-existent or a small group of symmetries at a generic point but getting enhanced to bigger groups at special points. On the other hand the spectra of the black hole ground states at different points in this moduli space are expected to be isomorphic since they represent the BPS states carrying a given set of charges. It should in principle be possible to use these isomorphisms to represent the action of the discrete symmetries at different points in the moduli space on the same Hilbert space. Thus all these discrete symmetries must be embedded in the single $U(N)$ group that acts on the N degenerate ground states of the black hole.

To take a concrete example, consider type IIA string theory compactified on $K3 \times T^2$ and take a black hole that carries only fundamental string winding, momentum, Kaluza-Klein (KK) monopole and H-monopole charges associated with the two circles of T^2. Although the full moduli space of the theory is parametrized locally by the $SO(6,22)/SO(6) \times SO(22)$ coset space, some of the moduli are fixed in the near horizon geometry leaving behind only a locally $SO(4,22)/SO(4) \times SO(22)$ space. The moduli labelling this space include in particular the metric and the 2-form fields on $K3$. Now it has been recently speculated in [71–78] that the symmetry group of supersymmetric states in a sigma model with target space $K3$ includes the Matthew group M_{24} even though there is no known point in the $K3$ moduli space at which the corresponding string theory has manifest M_{24} symmetry. In this case this group must also have a natural action on the space of BPS states of the black hole described above and sit inside the $U(N)$ group acting on the N degenerate ground states of the black hole. A better understanding of why M_{24} appears as a symmetry of supersymmetric states in the sigma model could help us realize this as an explicit symmetry of string theory in this particular near horizon geometry. This will still fall short of realizing the whole $U(N)$ group as a manifest symmetry, but will help us realize a large subgroup of $U(N)$.

For special values of the charges the symmetry group may also include a subgroup of the duality group associated with compactification on T^2. Note however that none of these symmetries may be a symmetry of the asymptotic theory since they can be broken by the expectation values of the moduli fields at infinity.

3. Supergravity theories reduced to two dimensions typically have a large group of continuous duality symmetries. Normally in the presence of charged particles this symmetry breaks down to a discrete subgroup. However since in the AdS_2 geometry there are no

charged excitations one could wonder if these continuous duality symmetries could play any role in building up the $U(N)$ group. In this context it is encouraging to note that the enhanced discrete symmetries at special points in the moduli space are naturally embedded in this continuous duality group.

Acknowledgement: I would like to thank Rajesh Gopakumar, Dileep Jatkar, Hong Liu, Shiraz Minwalla, Erik Tonni, Sandip Trivedi and Barton Zwiebach for discussion on various aspects of this work. This work was supported in part by the J. C. Bose fellowship of the Department of Science and Technology, India and the project 11-R& D-HRI-5.02-0304. I would also like to thank the Morningstar visiting professorship for support during a visit to Center for Theoretical Physics, MIT.

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